ON THE STABILITY OF MOTION OF A HEAVY GYROSCOPE ON GIMBALS

(OB USTOICHIVOSTI DVIZHENIA TIAZHELOGO GIROSKOPA V KARDANOVOM PODVESE)

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Chetaev [1] has investigated the motion of a heavy unbalanced gyroscope on gimbals with the vertical axis of the outer ring. The stability of certain stationary solutions of such a motion has been investigated by Skimel [2], Magnus [3] and Rumiantsev [4]. Similar investigations for the horizontal axis of the outer ring have been carried out by Rumiantsev [5]. Bogoiavlenskii [6] obtained in the latter case a particular integral of the equations of motion. This paper investigates the motion of a symmetric unbalanced gyroscope on gimbals when the axis of the outer ring makes any angle with the vertical [7], examines stability of certain stationary solutions and derives a particular solution when the axis of the outer ring is horizontal.

1. Let us introduce two fixed coordinate systems, $O\xi\eta\zeta$ and $Ox_1y_1z_1$, the origin O coinciding with the fixed point of the gyroscope. The ζ-axis is vertical, directed upwards, the ξ- and η-axes are in a horizontal plane. The z_1-axis is along the axis of the outer ring, and without any loss of generality we shall assume that the z_1-axis is in the $\eta\zeta$-plane making an angle a with the ζ-axis. The x_1-axis is also in the $\eta\zeta$-plane making an angle $(\pi/2 - a)$ with the ζ-axis. We introduce also a moving right-handed coordinate system $Oxyz$ attached to the inner ring, the x-axis being along the axis of the inner ring, the z-axis along the axis of the gyroscope.

The orientation of the gyroscope is determined by three angles, the rotation angle of the outer ring ψ, the rotation angle of the inner ring (casing) in the outer ring θ and the rotation angle of the gyroscope with respect to the $Oxyz$ system (angle of spin) ϕ.

Let l be the coordinate of the center of gravity of the gyroscope together with the inner ring (casing), m their mass, I the moment of inertia of the outer ring about the z_1-axis, A°, B° and C° the principal moments of inertia of the inner ring (casing) about the axes x, y and z, respectively, and A, $B = A$, C the moments of inertia of the gyroscope about the axes x, y and z, respectively. Using the above notation, the kinetic energy of the system (*vis viva*) is

$$2T = (A + A^\circ)\,\theta^{\cdot 2} + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi^{\cdot 2} + C\,(\varphi^\cdot + \psi^{\cdot 2}\cos\theta)^2$$

If we assume the absence of friction in the bearings and that the gravity force is the only active force, then for the force function we obtain the expression (correcting a certain inaccuracy contained in [7]):

$$U = -\,mgl\,(\sin\alpha\sin\theta\sin\psi + \cos\alpha\cos\theta)$$

As the coordinates θ, ψ and ϕ are independent and holonomic, we can write the equations of motion in the form of the Lagrange equations of the second kind. We have

$$\begin{aligned}
&(A + A^\circ)\,\theta^{\cdot\cdot} - (A + B^\circ - C^\circ)\,\psi^{\cdot 2}\sin\theta\cos\theta + C\,(\varphi^\cdot + \psi^\cdot\cos\theta)\,\psi^\cdot\sin\theta + \\
&\qquad\qquad + mgl\,(\sin\alpha\cos\theta\sin\psi - \cos\alpha\sin\theta) = 0 \\
&\frac{d}{dt}\{[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi^\cdot + C\,(\varphi^\cdot + \psi^\cdot\cos\theta)\cos\theta\} + mgl\sin\alpha\sin\theta\cos\psi = 0 \\
&\frac{d}{dt}[C\,(\varphi^\cdot + \psi^\cdot\cos\theta)] = 0 \qquad\qquad (1.1)
\end{aligned}$$

The above equations yield the two first integrals

$$\begin{aligned}
&(A + A^\circ)\,\theta^{\cdot 2} + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi^{\cdot 2} + C\,(\varphi^\cdot + \psi^\cdot\cos\theta)^2 + \\
&\quad + 2mgl\,(\sin\alpha\sin\theta\sin\psi + \cos\alpha\cos\theta) = 2h, \quad \varphi^\cdot + \psi^\cdot\cos\theta = r = \text{const}
\end{aligned}$$

The first of the two is the kinetic energy (*vis viva*) integral, which corresponds to the cyclic coordinate ϕ. If the axis of the outer ring is vertical ($\alpha > 0$) then Equations (1.1) have one more first integral

$$[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi^\cdot + C\,(\varphi^\cdot + \psi^\cdot\cos\theta)\cos\theta = k$$

corresponding to the angular coordinate ψ.

2. Let us investigate the stability of the particular solution

$$\theta = \alpha, \qquad \psi = \tfrac{1}{2}\pi, \qquad \theta^\cdot = 0, \qquad \psi^\cdot = 0, \qquad r = r_0 \qquad (2.1)$$

of Equations (1.1). In this case, the axis of the inner ring is horizontal, its middle plane is vertical and the gryoscope rotates with constant angular velocity r_0 about its vertical axis of spin. When $\alpha = \pi/2$ (the axis of the outer ring is vertical or horizontal) the corresponding solutions have been investigated in [4,5]. In what follows, the angle α will not be zero.

For the perturbed motion, we shall assume

$$\theta = \alpha + \xi_1, \quad \theta^{\cdot} = \xi_1^{\cdot} = \eta_1, \quad \psi = \tfrac{1}{2}\pi + \xi_2, \quad \psi^{\cdot} = \xi_2^{\cdot} = \eta_2, \quad r = r_0 + \eta_3$$

The equations of the perturbed motions yield the following integrals

$$V_1 = (A + A^\circ)\eta_1^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\alpha]\eta_2^2 + \\ + C\eta_3^2 + 2Cr_0\eta_3 - mgl(\xi_1^2 + \xi_2^2\sin^2\alpha) + \ldots = \mathrm{const}$$

$$V_2 = \eta_3 = \mathrm{const}$$

where the first of them is written up to the second-order terms inclusive. Let us consider the integral

$$V = V_1 - 2Cr_0V_2 = (A + A^\circ)\eta_1^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\alpha]\eta_2^2 + \\ + C\eta_3^2 - mgl(\xi_1^2 + \xi_2^2\sin^2\alpha) + \ldots = \mathrm{const}$$

When $l < 0$ and $\alpha \neq 0$, this integral becomes a sign-definite function of the variables ξ_1, ξ_2, η_1, η_2, η_3, hence, under our assumptions and on the strength of Liapunov's theorem on stability, the motion (2.1) is stable with respect to θ, ψ, $\dot\theta$, $\psi^{\cdot}$, r, and also with respect to $\theta^{\cdot}$, $\psi^{\cdot\cdot}$, r. Besides, the stability, as it can easily be demonstrated, is secular.

When $l > 0$, the function V is no longer sign-definite and the degree of instability is even. Therefore, the solution (2.1) could have only gyroscopic stability in this case. Let us find the condition for the latter kind of stability. The equations of the first approximation have the following form:

$$a\xi_1^{\cdot\cdot} = d\xi_1 - c\xi_2^{\cdot}, \qquad b\xi_2^{\cdot\cdot} = e\xi_2 + c\xi_1^{\cdot} \tag{2.2}$$

and yield the first integral, as given by Chetaev in [8]

$$\Gamma = 2(bd\,\xi_1\xi_2^{\cdot} - ae\xi_1^{\cdot}\xi_2) - c(d\xi_1^2 + e\xi_2^2) + \frac{ae - bd}{2c}(b\xi_2^{\cdot 2} - a\xi_1^{\cdot 2} + d\xi_1^2 - e\xi_2^2) = \mathrm{const}$$

Here

$$A + A^\circ = a, \qquad I + C^\circ + (A + B^\circ - C^\circ)\sin^2\alpha = b$$
$$Cr_0\sin\alpha = c, \qquad mgl = d, \qquad mgl\sin^2\alpha = e \qquad (r_0 \neq 0).$$

Let us consider the second integral

$$\frac{c}{2}V_1 - \Gamma - Ccr_0V_2 = \frac{c^2 - bd + ae}{2c}a\xi_1^{\cdot 2} + 2ae\xi_1^{\cdot}\xi_2 + \frac{c^2 - bd + ae}{2c}e\xi_2^2 + \\ + \frac{c^2 + bd - ae}{2c}b\xi_2^{\cdot 2} - 2bd\xi_2^{\cdot}\xi_1 + \frac{c^2 + bd - ae}{2c}d\xi_1^2 + \frac{1}{2}Cc\eta_3^2$$

When $l > 0$, this integral becomes sign-definite if the following single condition is satisfied:

$$c^2 - (ae + bd + 2\sqrt{abde}) > 0 \tag{2.3}$$

Hence, if the integral Γ is a continuation of the integral of the new equations of the perturbed motion, the condition (2.3) becomes the sufficient condition for stability of motion (2.1) with respect to θ, ψ, θ, ψ, r. Using the original notation, the condition (2.3) becomes

$$C^2 r_0^2 \sin^2\alpha - mgl\,[I + C^\circ + (2A + A^\circ + B^\circ - C^\circ)\sin^2\alpha +$$
$$+ 2\sin\alpha\sqrt{(A + A^\circ)\,[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\alpha]}\,] > 0$$

and for the case when the axis of the outer ring is horizontal, ($\alpha = \pi/2$), it coincides with the condition, obtained under corresponding assumptions in [5].

We shall demonstrate the necessity of the condition (2.3). To achieve this, we shall examine the characteristic equation of Equations (2.2) in variations. It has the following form:

$$ab\lambda^4 + (c^2 - ae - bd)\,\lambda^2 + de = 0 \tag{2.4}$$

If the motion (2.1) were stable, then all the roots of the above equation would have negative real parts. If $l > 0$ and $\alpha \neq 0$, then Equation (2.4) cannot have negative roots, and the roots would be pure imaginaries, with negative squares. The negative squares of the roots require the following inequalities

$$c^2 - ae - bd > 0, \qquad (c^2 - ae - bd)^2 - 4abde > 0$$

to be satisfied, which in turn requires satisfaction of (2.3). Thus, (2.3) becomes the necessary condition for stability of the motion (2.1) when $\alpha \neq 0$. For the case $\alpha = \pi/2$, the necessity of the condition (2.3) was shown by Chzhan Sy-in [9].

3. Let us consider one more particular solution of Equation (1.1):

$$\theta = 0, \qquad \psi = 0, \qquad \theta^{\cdot} = 0, \qquad \psi^{\cdot} = 0, \qquad r = r_0 \tag{3.1}$$

In this case, the middle planes of the outer and the inner rings coincide and are vertical, and the axis of spin coincides with the axis of the outer ring.

In the perturbed motion we shall set

$$\theta = \xi_1, \qquad \theta^{\cdot} = \xi_1^{\cdot}, \qquad \psi = \xi_2, \qquad \psi^{\cdot} = \xi_2^{\cdot}, \qquad r = r_0 + \eta_1$$

The equations of the perturbed motion written up to the second-order terms inclusive are

$$(A + A^\circ)\,\xi_1^{\cdot\cdot} - mgl\xi_1\cos\alpha + mgl\xi_2\sin\alpha + Cr_0\xi_1\xi_2^{\cdot} + \ldots = 0$$
$$(I + C^\circ)\,\xi_2^{\cdot\cdot} + mgl\xi_1\sin\alpha + C\eta_1^{\cdot} - Cr_0\xi_1\xi_1^{\cdot} + \ldots = 0, \quad \eta_1^{\cdot} = 0$$

The characteristic equation of the first approximation can be reduced to the form

$$\lambda^4 - \frac{mgl\cos\alpha}{A+A^\circ}\lambda^2 - \frac{(mgl\sin\alpha)^2}{(A+A^\circ)(I+C^\circ)} = 0$$

Whence

$$\lambda^2 = \frac{mgl}{2(A+A^\circ)}\left[\cos\alpha \pm \sqrt{\cos^2\alpha + \frac{4(A+A^\circ)}{I+C^\circ}\sin^2\alpha}\right]$$

In our case of unbalanced gyroscope ($l \neq 0$), the above formula shows that when α is in the interval $0 < \alpha < \pi/2$, the characteristic equation has one positive root. Hence, the motion (3.1) is unstable (on the strength of Liapunov's theorem on instability in the first approximation). From these considerations follows the instability of motion (3.1) at $\alpha = \pi/2$ and also in the case $l > 0$ and $\alpha = 0$.

4. In a special case, when the axis of the outer ring is horizontal ($\alpha = \pi/2$) it can be easily shown that Equations (1.1) have the following particular solution:

$$\psi = 0, \quad \psi^\cdot = 0, \quad \theta = \frac{mgl}{Cr_0}t + \theta_0, \quad \theta^\cdot = \frac{mgl}{Cr_0}, \quad \varphi = r_0 t + \varphi_0, \quad \varphi^\cdot = r_0 \qquad (4.1)$$

The middle planes of the outer and the inner rings are vertical in this case, the middle plane of the inner ring rotates with constant angular velocity about its vertical axis and the gyroscope spins also with constant angular velocity about its axis of spin. Thereby, the parameters of the system are related to the angular velocities through the formula $C\phi\theta = mgl$. The motion (4.1) has a character of regular precession with the nutation angle equalling $\pi/2$.

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